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APPLICATION OF REDUCED DIFFERENTIAL TRANSFORMATION METHOD FOR SOLVING SYSTEM OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS (PDES)

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ABSTRACT

In this paper, the reduced differential transformation method is used to obtain the solution of systems of nonlinear partial differential equation. The exact solutions of three systems of nonlinear partial differential equations are calculated in the form of series with easily computable components.

A comparison of the technique with some other known techniques like Adomian Decomposition Method (ADM), Variation Iteration Method (VIM) shows the simplicity, effectiveness and efficiency of the present approach with less computational work.

KEYWORDS: ADM, Nonlinear system, Partial Differential Equations, RDTM, VIM

I. INTRODUCTION

Partial Differential Equations (PDEs) have numerous applications in various fields of science and engineering such as fluid mechanic, thermodynamic, heat transfer and physics. (Debnath, 1997)

Systems of PDEs, linear or nonlinear have attracted much concern in studying evolution equations that describe wave propagation, in investigating shallow water waves and in examining the chemical reaction-diffusion model of Brusselator. Specifically, systems of nonlinear PDEs have also been noticed to arise in chemical and biological applications. The general ideas and the essential features of these systems are of wide applicability.

Several traditional methods such as method of characteristics and variational principle are among the methods that are useful in handling nonlinear PDEs. The existing techniques encountered some difficulties in terms of the size of computational work needed especially when the system involves several partial differential equations. (Wazwaz, 2009)

To avoid the difficulties that usually arise from traditional strategies, the Reduced differential transform method (Keskin and Oturanc, 2009) form a reasonable basis for studying systems of partial differential equations.

The method, as would be seen later, has a useful attraction in that solution is presented in a rapidly convergent power series with easily computable components.

2. BASIC IDEAS OF REDUCED DIFFERENTIAL TRANSFORM METHOD

Suppose that $u(x,t)$ is a function of two variables which is analytic and $k-times$ continuously differentiable with respect to time t and space x in our domain of interest.

Assume we can represent this function as a product of two single variable functions $u(x,t) = f(x)g(t)$.

From the definitions of differential transform method, the function can be represented as

$$u(x,t)=\sum_{i=0}^{\infty}F(i)x^i.\sum_{j=0}^{\infty}G(j)t^j=\sum_{k=0}^{\infty}U_k(x)t^k \qquad (2.1)$$

Where $U_k(x)$ is the transformed function, which can be defined as

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \qquad (2.2)$$

Thus from equations (2.1) and (2.2), we can deduce

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}t^k \qquad (2.3)$$

Considering equations $(2.1)-(2.3)$, it is clear that the concept of the RDTM is derived from the power series expansion.

The summary of the fundamental transformation properties of RDTM are shown in the table below:

Table 1: Basic Transformations of RDTM

Functional form	*Transformed form*
$u(x,t)$	$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t)=u(x,t)\pm v(x,t)$	$W_k(x)=U_k(x)\pm V_k(x)$
$w(x,t)=\lambda u(x,t)$	$W_k(x)=\lambda U_k(x)$
$w(x,t)=u(x,t)v(x,t)$	$W_k(x)=\sum_{i=0}^{k}U_i(x)V_{k-i}(x)=\sum_{i=0}^{k}V_i(x)U_{k-i}(x)$
$w(x,t)=p(x,t)q(x,t)r(x,t)$	$W_k(x)=\sum_{i=0}^{k}\sum_{j=0}^{i}P_j(x)Q_{i-j}(x)R_{k-i}(x)$
$w(x,t)=x^m t^n$	$W_k(x)=x^m\delta(k-n), \delta(k-n)=\begin{cases}1, k=n\\ 0, k\neq n.\end{cases}$
$w(x,t)=x^m t^n u(x,t)$	$W_k(x)=x^m U_{k-n}(x)$
$w(x,t)=\frac{\partial^n}{\partial t^n}u(x,t)$	$W_k(x)=\frac{(k+n)!}{k!}U_{k+n}(x)$
$w(x,t)=\frac{\partial^n}{\partial x^n}u(x,t)$	$W_k(x)=\frac{\partial^n}{\partial x^n}U_k(x)$
$w(x,t)=\frac{\partial^{n+m}}{\partial x^n\partial t^m}u(x,t)$	$W_k(x)=\frac{\partial^n}{\partial x^n}\frac{(k+m)!}{k!}U_{k+m}(x)$

3. APPLICATIONS

In this section, we apply the RDTM to three numerical examples of system of nonlinear partial differential equations and then compare our approximate solutions to the exact solutions.

Example 3.1: Consider the Nonlinear system of partial differential equations [7]

$$\begin{aligned} &u_t + vu_x + u - 1 = 0 \\ &v_t - uv_x - v - 1 = 0 \end{aligned} \qquad (3.1)$$

Subject to the initial conditions

$$\begin{aligned} &u(x,0) = e^{x} \\ &v(x,0) = e^{-x} \end{aligned} \qquad (3.2)$$

Applying the basic properties of the RDTM, we obtain the transformed form of equation (3.1) as

$$(k+1)U_{k+1}(x) = -\sum_{i=0}^{k} V_i(x)\frac{\partial}{\partial x}U_{k-i}(x) + N_k(x)$$

$$(k+1)V_{k+1}(x) = \sum_{i=0}^{k} U_i(x)\frac{\partial}{\partial x}V_{k-i}(x) + N_k{}'(x)$$

i.e

$$\begin{aligned} &U_{k+1}(x) = \frac{1}{k+1}\left(-\sum_{i=0}^{k} V_i(x)\frac{\partial}{\partial x}U_{k-i}(x) + N_k(x)\right) \\ &V_{k+1}(x) = \frac{1}{k+1}\left(\sum_{i=0}^{k} U_i(x)\frac{\partial}{\partial x}V_{k-i}(x) + N_k{}'(x)\right) \end{aligned} \qquad (3.3)$$

where $N_k(x)$ is the transformed form of $1-u$ and $N_k{}'$ is the transformed form of $v+1$.

Easily the first few nonlinear terms are:

$$\begin{aligned} &N_0 = 1 - u_0 \\ &N_1 = -u_1 \\ &N_2 = -u_2, \ldots \end{aligned} \quad \text{and} \quad \begin{aligned} &N_0{}' = v_0 + 1 \\ &N_1{}' = v_1 \\ &N_2{}' = v_2, \ldots \end{aligned}$$

Using the initial condition (3.2), we have

$$\begin{aligned} &u_0 = e^{x} \\ &v_0 = e^{-x} \end{aligned} \qquad (3.4)$$

Now substituting (3.4) into (3.3), we obtain the following $U_k(x)$, $V_k(x)$ values successively;

$$U_1(x) = -e^x \qquad V_1(x) = e^{-x}$$
$$U_2(x) = \frac{1}{2}e^x \qquad V_2(x) = \frac{1}{2}e^{-x}$$
$$U_3(x) = -\frac{1}{6}e^x \quad \text{and} \quad V_3(x) = \frac{1}{6}e^{-x}$$
$$U_4(x) = \frac{1}{24}e^x \qquad V_4(x) = \frac{1}{24}e^{-x}$$
$$\ldots \qquad \ldots$$
$$U_n(x) = \frac{(-1)^n}{n!}e^x \qquad V_n(x) = \frac{1}{n!}e^{-x}$$

Finally, the differential inverse transform of $u_k(x),\ v_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k e^x = e^x - te^x + \frac{t^2}{2!}e^x - \frac{t^3}{3!}e^x + \ldots = e^{x-t}$$

And

$$v(x,t) = \sum_{k=0}^{\infty} V_k(x)t^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} e^{-x} = e^{-x} + te^{-x} + \frac{t^2}{2!}e^{-x} + \frac{t^3}{3!}e^{-x} + \ldots = e^{-x+t}$$

Which is the same as the exact solution of the system of nonlinear partial differential equations $(3.1)-(3.2)$ given by $u(x,t), v(x,t) = e^{x-t}, e^{-x+t}$.

Example 3.2: Consider the nonlinear system of partial differential equation [7]

$$\begin{aligned} u_t + v_x w_y - v_y w_x &= -u \\ v_t + w_x u_y + w_y u_x &= v \\ w_t + u_x v_y + u_y v_x &= w \end{aligned} \qquad (3.5)$$

Subject to the initial conditions

$$\begin{aligned} u(x,y,0) &= e^{x+y} \\ v(x,y,0) &= e^{x-y} \\ w(x,y,0) &= e^{-x+y} \end{aligned} \qquad (3.6)$$

Applying the basic properties of the RDTM, we obtain the transformed form of equation (3.5)

$$(k+1)U_{k+1}(x) = -\sum_{i=0}^{k} \frac{\partial}{\partial x} V_i(x) \frac{\partial}{\partial y} W_{k-i}(y) + \sum_{i=0}^{k} \frac{\partial}{\partial y} V_i(y) \frac{\partial}{\partial x} W_{k-i}(x) - U_k(x,y)$$
$$(k+1)V_{k+1}(x) = -\sum_{i=0}^{k} \frac{\partial}{\partial x} W_i(x) \frac{\partial}{\partial y} U_{k-i}(y) - \sum_{i=0}^{k} \frac{\partial}{\partial y} W_i(y) \frac{\partial}{\partial x} U_{k-i}(x) + V_k(x,y)$$
$$(k+1)W_{k+1}(x) = -\sum_{i=0}^{k} \frac{\partial}{\partial x} U_i(x) \frac{\partial}{\partial y} V_{k-i}(y) - \sum_{i=0}^{k} \frac{\partial}{\partial y} U_i(y) \frac{\partial}{\partial x} V_{k-i}(x) + W_k(x,y)$$

i.e.

$$U_{k+1}(x) = \left(\frac{-1}{k+1}\right)\left[\sum_{i=0}^{k} \frac{\partial}{\partial x} V_i(x) \frac{\partial}{\partial y} W_{k-i}(y) + \sum_{i=0}^{k} \frac{\partial}{\partial y} V_i(y) \frac{\partial}{\partial x} W_{k-i}(x) - U_k(x,y) \right]$$
$$V_{k+1}(x) = \left(\frac{-1}{k+1}\right)\left[\sum_{i=0}^{k} \frac{\partial}{\partial x} W_i(x) \frac{\partial}{\partial y} U_{k-i}(y) - \sum_{i=0}^{k} \frac{\partial}{\partial y} W_i(y) \frac{\partial}{\partial x} U_{k-i}(x) + V_k(x,y) \right]$$
$$W_{k+1}(x) = \left(\frac{-1}{k+1}\right)\left[\sum_{i=0}^{k} \frac{\partial}{\partial x} U_i(x) \frac{\partial}{\partial y} V_{k-i}(y) - \sum_{i=0}^{k} \frac{\partial}{\partial y} U_i(y) \frac{\partial}{\partial x} V_{k-i}(x) + W_k(x,y) \right]$$

(3.7)

Using the initial condition (3.6), we have

$$u_0 = e^{x+y}$$
$$v_0 = e^{x-y} \qquad (3.8)$$
$$w_0 = e^{-x+y}$$

Now, substituting equation (3.8) into equation (3.7), we obtain the following $U_k(x,y)$, $V_k(x,y)$ and $W_k(x,y)$ values successively

$$U_1(x,y) = -e^{x+y} \qquad V_1(x,y) = e^{x-y} \qquad W_1(x,y) = e^{-x+y}$$
$$U_2(x,y) = \frac{1}{2}e^{x+y} \qquad V_2(x,y) = \frac{1}{2}e^{x-y} \qquad W_2(x,y) = \frac{1}{2}e^{-x+y}$$
$$U_3(x,y) = -\frac{1}{6}e^{x+y}, \qquad V_3(x,y) = \frac{1}{6}e^{x-y} \quad \text{and} \quad W_3(x,y) = \frac{1}{6}e^{-x+y}$$
$$U_4(x,y) = \frac{1}{24}e^{x+y} \qquad V_4(x,y) = \frac{1}{24}e^{x-y} \qquad W_4(x,y) = \frac{1}{24}e^{-x+y}$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$U_n(x,y) = \frac{(-1)^n}{n!}e^{x+y} \qquad V_n(x,y) = \frac{1}{n!}e^{x-y} \qquad W_n(x,y) = \frac{1}{n!}e^{-x+y}$$

Finally, the differential inverse transforms of $U_k(x,y)$, $V_k(x,y)$ and $W_k(x,y)$ gives

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^k=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^k e^{x+y}$$
$$u(x,y,t)=e^{x+y}-te^{x+y}+\frac{t^2}{2!}e^{x+y}-\frac{t^3}{3!}e^{x+y}+\ldots=e^{x+y-t}$$

$$v(x,y,t)=\sum_{k=0}^{\infty}V_k(x,y)t^k=\sum_{k=0}^{\infty}\frac{1}{k!}t^k e^{x-y}$$
$$v(x,y,t)=e^{x-y}+te^{x-y}+\frac{t^2}{2!}e^{x-y}+\frac{t^3}{3!}e^{x-y}+\ldots=e^{x-y-t}$$

And

$$w(x,y,t)=\sum_{k=0}^{\infty}W_k(x,y)t^k=\sum_{k=0}^{\infty}\frac{1}{k!}t^k e^{-x+y}$$
$$w(x,y,t)=e^{-x+y}+te^{-x+y}+\frac{t^2}{2!}e^{-x+y}+\frac{t^3}{3!}e^{-x+y}+\ldots=e^{-x+y-t}$$

Which is the same as the exact solution of the system of nonlinear partial differential equations (3.6)-(3.7) given by

$$u(x,y,t),v(x,y,t),w(x,y,t)=e^{x+y-t},e^{x-y-t},e^{-x+y-t}.$$

Example 3.3: Consider the nonlinear system of partial differential equation [7]

$$\begin{aligned} u_t+u_y v_x &= 1+e^t \\ v_t+v_y w_x &= 1-e^{-t} \\ w_t+w_y u_y &= 1-e^{-t} \end{aligned} \quad (3.9)$$

Subject to the initial conditions

$$\begin{aligned} u(x,y,0) &= 1+x+y \\ v(x,y,0) &= 1+x-y \\ w(x,y,0) &= 1-x+y \end{aligned} \quad (3.10)$$

Applying the basic properties of the RDTM, we obtain the transformed form of equation (3.9)

$$(k+1)U_{k+1}(x)=-\sum_{i=0}^{k}\frac{\partial}{\partial y}U_i(y)\frac{\partial}{\partial x}V_{k-i}(x)+N_k(x)$$
$$(k+1)V_{k+1}(x)=-\sum_{i=0}^{k}\frac{\partial}{\partial y}V_i(y)\frac{\partial}{\partial x}W_{k-i}(x)+F_k(x)$$
$$(k+1)W_{k+1}(x)=-\sum_{i=0}^{k}\frac{\partial}{\partial y}W_i(y)\frac{\partial}{\partial y}U_{k-i}(y)+F_k(x)$$

i.e.

$$U_{k+1}(x)=\left(\frac{-1}{k+1}\right)\left[\sum_{i=0}^{k}\frac{\partial}{\partial y}U_i(y)\frac{\partial}{\partial x}V_{k-i}(x)-N_k(x)\right]$$
$$V_{k+1}(x)=\left(\frac{-1}{k+1}\right)\left[\sum_{i=0}^{k}\frac{\partial}{\partial y}V_i(y)\frac{\partial}{\partial x}W_{k-i}(x)-F_k(x)\right]$$
$$W_{k+1}(x)=\left(\frac{-1}{k+1}\right)\left[\sum_{i=0}^{k}\frac{\partial}{\partial y}W_i(y)\frac{\partial}{\partial y}U_{k-i}(y)-F_k(x)\right]$$

$N_k(x)$ is the transformed form of $1+e^t$ and $F_k(x)$ is the transformed form of $1-e^{-t}$

Easily the first few nonlinear terms are:

$$\begin{aligned} N' &= e^t & \qquad F' &= e^{-t} \\ N'' &= e^t & \text{and} \quad F'' &= -e^{-t} \\ N''' &= e^t,\ldots & F''' &= e^{-t},\ldots \end{aligned} \tag{3.11}$$

Using the initial condition (3.10), we have

$$\begin{aligned} u_0 &= 1+x+y \\ v_0 &= 1+x-y \\ w_0 &= 1-x+y \end{aligned} \tag{3.12}$$

Now, substituting equation (3.12) into equation (3.11), we obtain the following $U_k(x,y)$, $V_k(x,y)$ and $W_k(x,y)$ values successively

$$\begin{aligned} &U_1(x,y)=1 && V_1(x,y)=-1 && W_1(x,y)=-1 \\ &U_2(x,y)=\frac{1}{2} && V_2(x,y)=\frac{1}{2} && W_2(x,y)=\frac{1}{2} \\ &U_3(x,y)=\frac{1}{3}, && V_3(x,y)=-\frac{1}{3} \quad \text{and} && W_3(x,y)=-\frac{1}{3} \\ &U_4(x,y)=\frac{1}{4} && V_4(x,y)=\frac{1}{4} && W_4(x,y)=\frac{1}{4} \\ &\ldots && \ldots && \ldots \\ &U_n(x,y)=\frac{1}{n} && V_n(x,y)=\frac{(-1)^n}{n} && W_n(x,y)=\frac{(-1)^n}{n} \end{aligned}$$

Finally, the differential inverse transforms of $U_k(x,y)$, $V_k(x,y)$ and $W_k(x,y)$ gives

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^k=1+x+y+t+\frac{1}{2}t^2+\ldots$$
$$u(x,y,t)=x+y+e^t$$

$$v(x,y,t)=\sum_{k=0}^{\infty}V_k(x,y)t^k=1+x-y-t+\frac{1}{2}t^2-\frac{1}{3}t^3+\ldots$$
$$v(x,y,t)=x-y+e^{-t}$$

And

$$w(x,y,t)=\sum_{k=0}^{\infty}W_k(x,y)t^k=1-x+y-t+\frac{1}{2}t^2-\frac{1}{3}t^3+...$$
$$w(x,y,t)=-x+y+e^{-t}$$

Which is the same as the exact solution of the system of nonlinear partial differential equations (3.9)-(3.10) given by

$$u(x,y,t),v(x,y,t),w(x,y,t)=x+y+e^{t},x-y+e^{-t},-x+y+e^{-t}.$$

4. CONCLUSIONS

In this paper, the RDTM was used to obtain the solution of three systems of nonlinear partial differential equations.

The RDTM is a direct method which does not require any discretization and it approaches the exact solution rapidly, unlike the ADM and VIM which requires the computation of Adomian polynomials and correction functional respectively.

We thus conclude that the method is a very powerful one, which from this study gives the exact solution of nonlinear system of partial differential equations in a simple way with less computational work as compared to applications of ADM and VIM to nonlinear system of partial differential equations.

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